Predictability of Population Fluctuations

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Abstract

Population dynamics is affected by environmental fluctuations (such as climate variations), which have a characteristic correlation time. Strikingly, the time scale of predictability can be larger for the population dynamics than for the underlying environmental fluctuations. Here, we present a general mechanism leading to this increase in predictability. We considered colored environmental fluctuations acting on a population close to equilibrium. In this framework, we derived the temporal auto and cross-correlation functions for the environmental and population fluctuations. We found a general correlation time hierarchy led by the environmental-population correlation time, closely followed by the population autocorrelation time. The increased predictability of the population fluctuations arises as an increase in its autocorrelation and cross-correlation times. These increases are enhanced by the slow damping of the population fluctuations, which has an integrative effect on the impact of correlated environmental fluctuations. Therefore, population fluctuations predictability is enhanced when the damping time of the population fluctuation is larger than the correlation time of the environmental fluctuations. This general mechanism can be quite frequent in nature, and it largely increases the perspectives of making reliable predictions of population fluctuations.

Model: One species with temporally correlated noise

The relative **fluctuation from equilibrium** (N_{eq}) of a **population** of **size** $N(t)$ is defined as $\boldsymbol{\varepsilon(t)} = \frac{\boldsymbol{N(t)-N_{eq}}}{\boldsymbol{N_{eq}}}$. Evolution of population fluctuations is described by

$$\frac{d\varepsilon}{dt} = -\frac{\varepsilon}{T} + \lambda\, A. \qquad (1)$$

$\boldsymbol{T}$ is the **characteristic time of return to equilibrium**, $\boldsymbol{A}$ represents the **environmental fluctuations**, and $\boldsymbol{\lambda}$ is a **coupling constant**. We considered **temporally autocorrelated environmental fluctuations** with **amplitude** $\boldsymbol{\sigma}$ and a **characteristic correlation time** $\boldsymbol{\tau}$, described by

$$\frac{dA}{dt} = -\frac{A}{\tau} + \frac{\sigma}{\tau}\xi, \qquad (2)$$

where ξ represents a **normalized white noise** with zero mean ($<\xi>=0$).

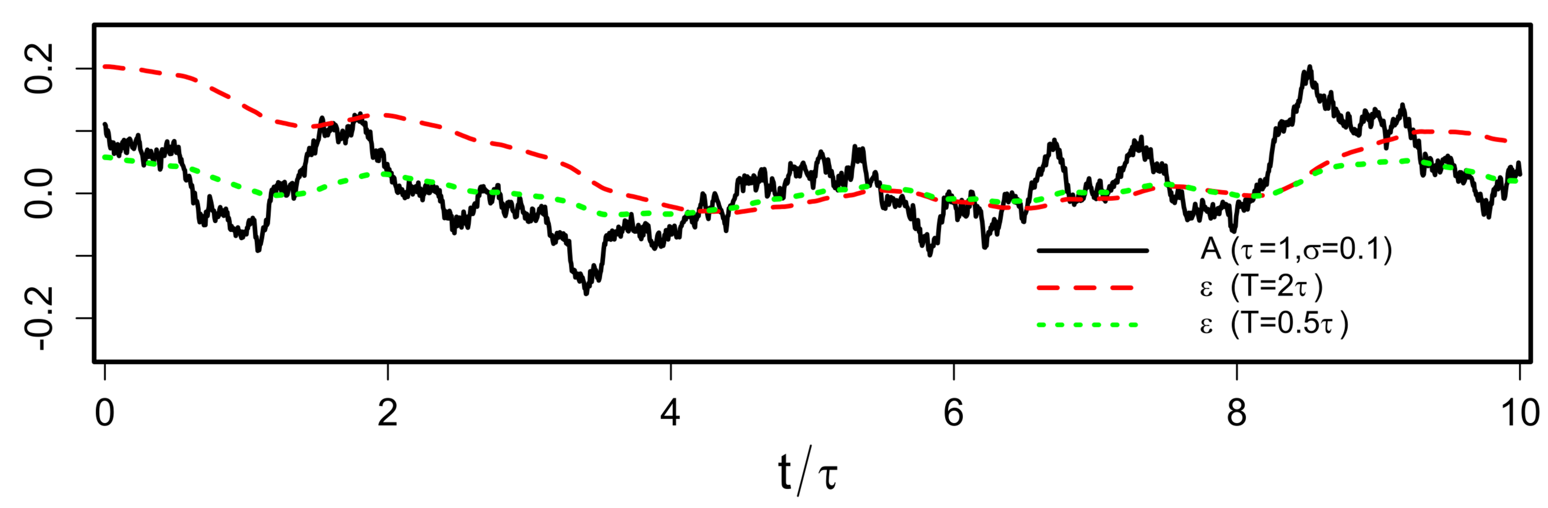

Fig 1. *Evolution for the environmental fluctuations A (solid black line); and the population fluctuations for $T=2\tau$ (red dashed line), and $T=0.5\tau$ (green pointed line) for $\sigma=0.1$, $\lambda=1$ and $\tau=1$.*

Temporal autocorrelations and cross-correlations

The **correlation function** between two magnitudes X and Y at time delay t' is

$$c_{XY}(t') = < X(t)Y(t+t') >, \qquad (3)$$

with <> the expected value. Defining $\alpha = T/\tau$ we get:

$$c_{AA}(t') = \frac{\sigma^2}{2\tau} e^{-|t'|/\tau} \qquad (4)$$

$$c_{\varepsilon\varepsilon}(t') = \begin{cases} \frac{\lambda^2\sigma^2\tau}{2}\frac{\alpha^2}{1-\alpha^2}\left(e^{-|t'|/\tau} - \alpha\, e^{-|t'|/T}\right), & T \neq \tau \\ \frac{\lambda^2\sigma^2\tau}{4}\left(1+|t'|/\tau\right)e^{-|t'|/\tau}, & T = \tau \end{cases} \qquad (5)$$

$$c_{A\varepsilon}(t') = c_{\varepsilon A}(-t') = \begin{cases} \frac{\lambda\sigma^2}{2}\frac{\alpha}{1+\alpha}e^{t'/\tau}, & t' \le 0 \\ \frac{\lambda\sigma^2}{2}\frac{\alpha}{1-\alpha^2}\left((1+\alpha)e^{-t'/\tau} - 2\alpha\, e^{-t'/T}\right), & t' > 0; T \neq \tau \\ \frac{\lambda\sigma^2}{4}(1+2t'/\tau)e^{-t'/\tau}, & t' > 0; T = \tau \end{cases} \qquad (6)$$

The value of the autocorrelation functions at $t'=0$ (equal to its maximum value) give the **variance of the fluctuations**:

$$M_{AA} = c_{AA}(0) = \frac{\sigma^2}{2\tau} \qquad (7)$$

$$M_{\varepsilon\varepsilon} = c_{\varepsilon\varepsilon}(0) = \frac{\lambda^2\sigma^2\tau}{2}\frac{\alpha^2}{1+\alpha} \qquad (8)$$

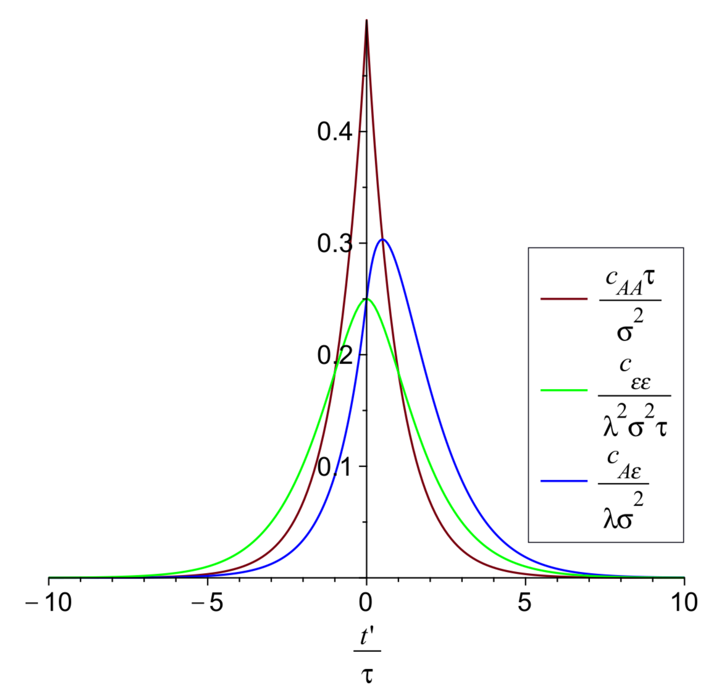

Fig 2. *Adimensionalized correlation functions $c_{\varepsilon\varepsilon}(t')$ (green), $c_{AA}(t')$ (red) and $c_{A\varepsilon}(t')$ (blue) for $\alpha = T/\tau = 1$.*

The **cross-correlation function** has a **lagged maximum** at time displacement $t' = l_{A\varepsilon}$, with normalized magnitude $M^N_{A\varepsilon} = \frac{M_{A\varepsilon}}{\sqrt{c_{AA}(0)c_{\varepsilon\varepsilon}(0)}}$

$$l_{A\varepsilon} = \begin{cases} \tau\frac{\alpha}{1-\alpha}\ln\left(\frac{2}{1+\alpha}\right), & T \neq \tau \\ \frac{\tau}{2}, & T = \tau \end{cases} \qquad (9)$$

$$M^N_{A\varepsilon} = \begin{cases} \frac{\lambda\,\sigma^2}{2}\,\alpha\left(\frac{2}{1+\alpha}\right)^{\frac{\alpha}{\alpha-1}}, & T \neq \tau \\ \frac{\lambda\,\sigma^2}{2}e^{-1/2}, & T = \tau \end{cases} \qquad (10)$$

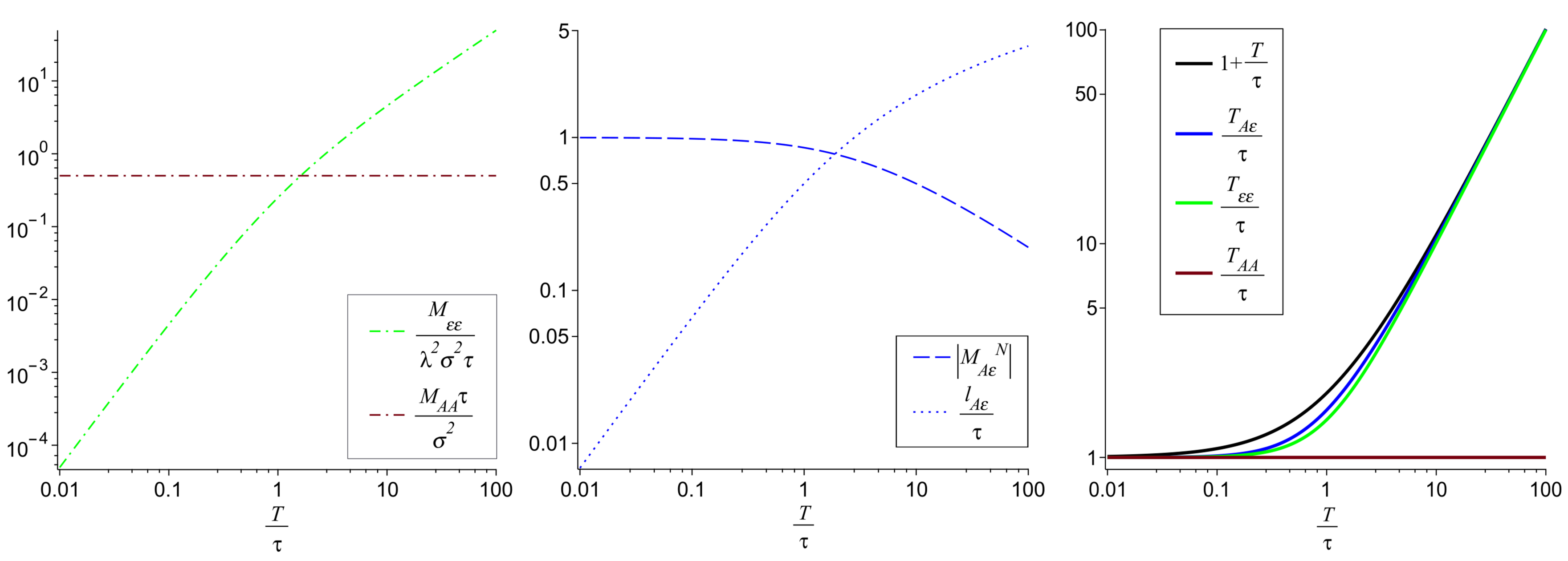

Fig 3. *(Left) Comparison between the maximum autocorrelation of the environmental fluctuations M_{AA} and of the population fluctuations $M_{\varepsilon\varepsilon}$. (Center) Normalized value of the maximum cross-correlation $M^N_{A\varepsilon}$ and its lag $l_{A\varepsilon}$. (Right) Comparison between the correlation times and the upper bound $\tau + T$.*

Correlation times

The **correlation time** measures the time extension of the predictability.

$$T_{XY} = \frac{\int_0^\infty t'|c_{XY}(t')|dt'}{\int_0^\infty |c_{XY}(t')|dt'}. \qquad (11)$$

For our autocorrelation and cross-correlation functions:

$$T_{AA} = T_{\varepsilon A} = \tau \;\; (12);$$

$$T_{\varepsilon\varepsilon} = \tau + T\frac{1}{1+1/\alpha} \;\; (13); \quad T_{A\varepsilon} = \tau + T\frac{1}{1+1/(2\alpha)}. \;\; (14)$$

The scale of predictability is always larger for the population fluctuations than that for the environmental fluctuations, as shown in the hierarchy

$$\boldsymbol{T_{AA} = T_{\varepsilon A} = \tau < T_{\varepsilon\varepsilon} < T_{A\varepsilon} < \tau + T} \qquad (15)$$

Conclusions and future work

We aimed to understand the predictability of population fluctuations and environmental fluctuations. We computed their auto and cross-correlation functions, finding a hierarchy for their correlation times, Eq. (15). This hierarchy stresses that the predictability of population fluctuations can be higher than that of environmental fluctuations. These results increase the hope to improve conservation policies for species strongly sensitive to environmental variability

This model can be extended by including several interacting species (in progress), or dividing the population into life stages. These models provide a tool to study the coupling between environmental fluctuations (e.g., SST anomalies) and ocean trophic levels in areas of interest for TRIATLAS project.

Reference

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