A Quantum Interior-Point Predictor-Corrector Algorithm for Linear Programming

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Linear Programming

We want to solve the two equivalent (dual) problems:

$$\min c^T x; \quad | \quad Ax \geq b, \quad x \geq 0. \tag{1}$$

$$\max b^T y; \quad | \quad A^T y \leq c. \tag{2}$$

Important in many areas such as planning, logistics, economics... We have the two main methods:

Figure 1: Simplex method (left) vs Interior Point (IP) Method (right).

Simplex method is simpler but may have exponential cost on the number of variables n.

Predictor-Corrector algorithm [1]

Objective: Combine (1) and (2) into a single problem. Define

$$\bar{b} := b - Ax^0, \qquad \bar{c} := c - A^T y^0 - s^0, \qquad \bar{z} := c^T x^0 + 1 - b^T y^0. \tag{3}$$

Solve the following self-dual problem: $\min \theta$ given

$$\begin{array}{rrrrl} & +Ax & -b\tau & +\bar{b}\theta & = 0 \\ -A^T y & & +c\tau & -\bar{c}\theta & \geq 0 \\ +b^T y & -c^T x & & +\bar{z}\theta & \geq 0 \\ -\bar{b}^T y & +\bar{c}^T x & -\bar{z}\tau & & = -(x^0)^T s^0 - 1 \end{array} \tag{4}$$

To do that consider $v^t = (y^t, x^t, \tau^t, \theta^t, s^t, k^t)$ and define the central path

$$\mathcal{N}(\beta) = \{(y,x,\tau,\theta,s,k) \in \mathcal{F}_h^0 : \left\| \begin{pmatrix} Xs \\ \tau k \end{pmatrix} - \mu 1_{(n+1)\times 1} \right\| \leq \beta\mu \text{ where } \mu = \frac{x^T s + \tau k}{n+1}\}. \tag{5}$$

Then, iteratively solve:

$$\begin{array}{c c} & \begin{array}{cccccc} m & n & 1 & 1 & n & 1 \end{array} \\ \begin{array}{c} m \\ n \\ 1 \\ 1 \\ n \\ 1 \end{array} & \begin{pmatrix} 0 & A & -b & \bar{b} & 0 & 0 \\ -A^T & 0 & c & -\bar{c} & -1 & 0 \\ b^T & -c^T & 0 & \bar{z} & 0 & -1 \\ -\bar{b}^T & \bar{c}^T & -\bar{z} & 0 & 0 & 0 \\ 0 & S^t & 0 & 0 & X^t & 0 \\ 0 & 0 & k^t & 0 & 0 & \tau^t \end{pmatrix} \end{array} \begin{pmatrix} d_y \\ d_x \\ d_\tau \\ d_\theta \\ d_s \\ d_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \gamma\mu^t 1_{n\times 1} - X^t s^t \\ \gamma\mu^t - \tau^t k^t \end{pmatrix}. \tag{6}$$

1. *Predictor step:* Solve (6) with $\gamma^t = 0$. Then find (bisection search) $\max\delta$ such that $v^{t+1} = v^t + \delta d_{v^t} \in \mathcal{N}(1/2)$ and sum it.
2. *Corrector step:* Solve (6) with $\gamma^t = 1$ and $v^{t+1} = v^t + d_{v^t} \in \mathcal{N}(1/4)$.

Linear System of Equations Algorithm [2]

A modification of the HHL algorithm for dense system of equations:

Theorem 1 [2]: *Let M be an $n' \times n'$ Hermitian matrix (if the matrix is not Hermitian it can be included as a submatrix of a Hermitian one) with condition number κ and Frobenius norm $||M||_F = \sqrt{\sum_{ij} M_{ij}^2}$. Let f be an n'-dimensional unit vector, and assume that there is an oracle $\mathcal{P}_f$ which produces the state $|f\rangle$. Let also M have spectral decomposition $M = \sum_i \lambda_i u_i u_i^\dagger$ encoded in a quantum accessible data structure (see theorem 2). Let*

$$d_v = M^{-1} f, \qquad |d\rangle = \frac{d_v}{||d_v||}. \tag{7}$$

Then, [2] constructs an algorithm relying on Quantum Singular Value Estimation [3] that outputs the state $|d\rangle$ up to precision ϵ^{-1}, with probability of failure $1 - 1/poly(n')$, and has overall time complexity

$$O(||M||_F(\kappa^2/\epsilon)\ \textit{poly}\log(n')). \tag{8}$$

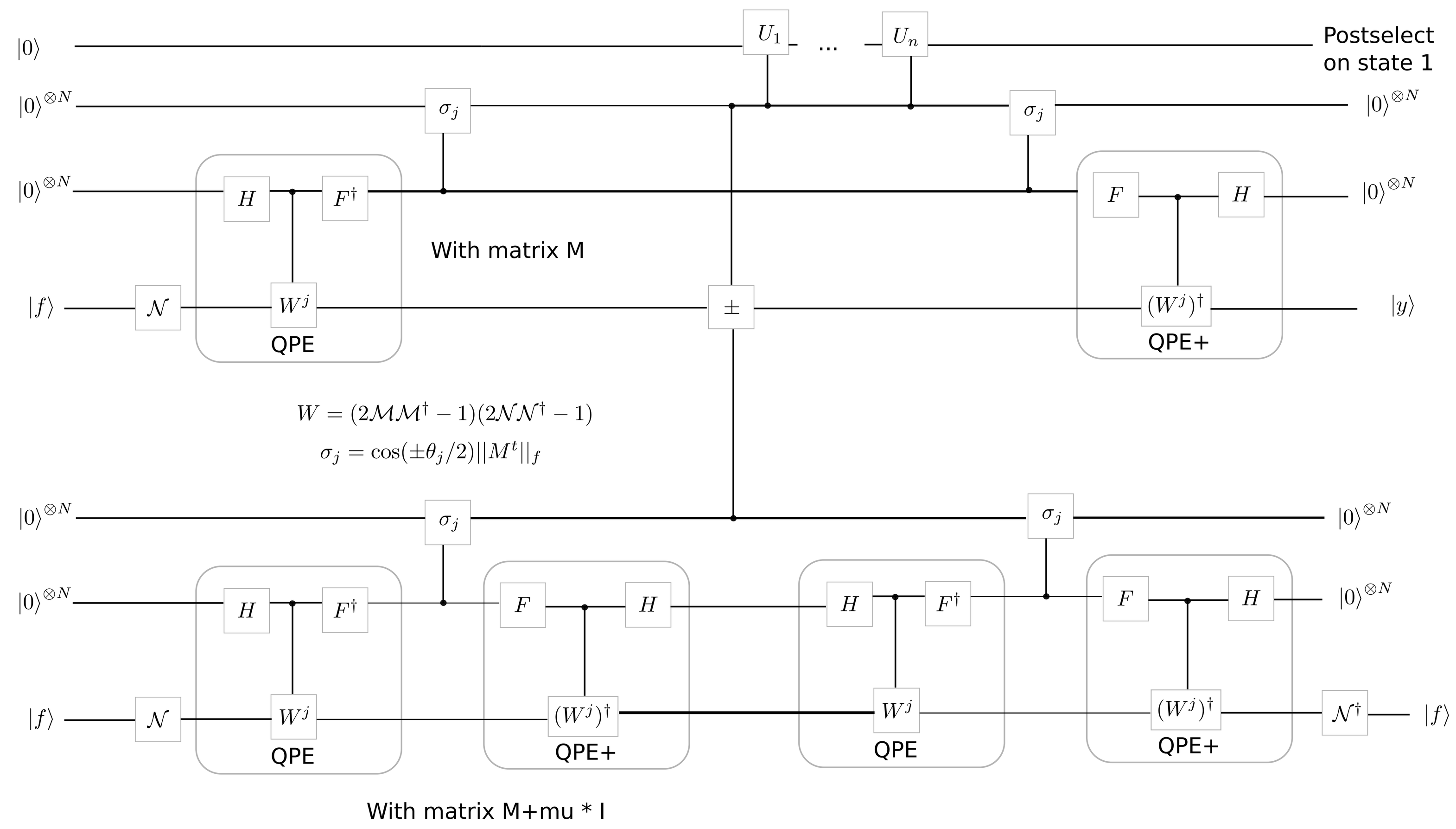

Figure 2: Circuit of the Dense version of the HHL algorithm.

Encoding

We need a fast encoding system. We prepare a classical data base that can be used to prepare states fast and also perform $U_\mathcal{M}$ and $U_\mathcal{N}$ from the Dense Quantum Linear System Algorithm:

Theorem 2 [4]: *Let $M \in \mathbb{R}^{n'\times n'}$ be a matrix. If w is the number of nonzero entries, there is a quantum accessible data structure of size $O(w\log^2(n'^2))$, which takes time $O(\log(n'^2))$ to store or update a single entry. Once the data structure is set up, there are quantum algorithms that can perform the following maps to precision ϵ^{-1} in time $O(\textit{poly}\log(n'^2/\epsilon))$:*

$$U_\mathcal{M} : |i\rangle|0\rangle \to \frac{1}{||M_{i\cdot}||}\sum_j M_{ij}|ij\rangle; \qquad U_\mathcal{N} : |0\rangle|j\rangle \to \frac{1}{||M||_F}\sum_i ||M_{i\cdot}||\,|ij\rangle; \tag{9}$$

where $||M_{i\cdot}||$ is the l_2-norm of row i of M. This means in particular that given a vector f in this data structure, we can prepare an ϵ approximation of it, $1/||v||_2 \sum_i v_i|i\rangle$, in time $O(\textit{poly}\log(n'/\epsilon))$.

Readout

To perform the readout of the linear system of equations we use Amplitude Estimation [5]. However, AE does not give sign, but only absolute value. To estimate relative sign between entries of the solution $|d\rangle$, calculate $|\langle d|R_{ij}\rangle|^2$ with $|R_{ij}\rangle := C_{ij}(|d_i|\,|j\rangle + |d_j|\,|i\rangle)$.

1. If same sign for entries i and j → The result is $|\langle d|R_{ij}\rangle|^2 = 2C_{ij}d_i d_j$.
2. If opposite sign for entries i and j → The result is $|\langle d|R_{ij}\rangle|^2 = 0$.

Error

The error in the Dense Quantum Linear System Algorithm is greater than for its classical counterpart: how do we ensure that we do not get out of $\mathcal{N}(1/4)$ in the corrector step? The answer is that if we get ϵ' out of $\mathcal{N}(1/4)$ due to error, we then perform an ϵ'-size step of gradient descent to go back in. That solves the problem.

Conclusion

Algorithms for Linear Programming	Work complexity
Multiplicative weights	$O((\sqrt{n}\left(\frac{Rr}{\epsilon}\right) + \sqrt{m})\left(\frac{Rr}{\epsilon}\right)^4)$
Another Quantum Interior Point Algorithm	$O(L\sqrt{n}(n+m)\mu\bar{\kappa}^3\epsilon^{-2})$;
Pred-Corr. + Conjugate Gradient	$O(L\sqrt{n}(n+m)^2\bar{\kappa}\log(\epsilon^{-1}))$
Pred-Corr. + Cholesky decomposition	$O(L\sqrt{n}(n+m)^3)$
Pred-Corr. + Optimal exact	$O(L\sqrt{n}(n+m)^{2.737})$
Pred-Corr. + QLSA (This algorithm)	$O(L\sqrt{n}(n+m)\overline{\lvert\lvert M\rvert\rvert_F}\bar{\kappa}^2\epsilon^{-2})$;

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