Extinction thresholds of spatially extended population with Allee effect

Rodrigo Crespo[a], Francisco J. Cao[a,b]

[a] Dpto. de Estructura de la Materia, Física Térmica y Electrónica, UCM; [b] Instituto Madrileño de Estudios Avanzados en Nanociencia (IMDEA-Nanociencia)

Abstract

Many species are unsustainable at small population densities (Allee Effect). This implies that for population densities below a threshold, named Allee threshold, the population decreases instead of growing. Here, we have studied how a ecosystem with a stable population density (close to carrying capacity) can become extinct due to large environmental fluctuations, which lead the population below the Allee threshold.

Spatially extended population model

The Allee population growth model with diffusion and environmental fluctuations is given by

$$dN = \left(-rN(N-A)(N-K) - mN + m\int N(x-y,t)f(y)dy \right) dt + \sigma N dB, \qquad (1)$$

where K is the carrying capacity (stable viable population), and A is the Allee threshold (minimum viable population). In this equation m is the diffusion rate and f(y) the diffusion profile, which has width l_m. Environmental fluctuations have amplitude σ and spatial correlation length l_e

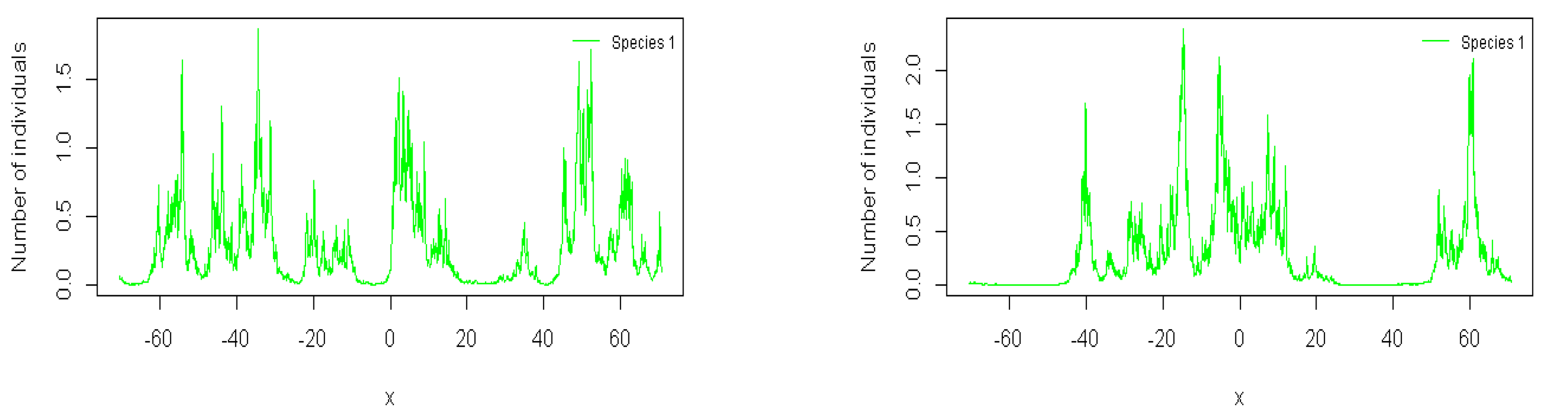

Fig 1. *Spatial population distribution N(x) at two times: Evolution with N_0 =K=1, A = 0.1, r=m=l_e=l_m = 1, and σ= 0.75.*

Population probability distribution and extinction threshold

The stationary population probability can be computed using the evolution equation has the form

$$dN = F(N)dt + \sqrt{v(N)}dB \quad (2)$$

which has the stationary population probability distribution

$$p(N) = n \cdot \exp\left(2\int \frac{F(N)}{v(N)} dN \right) \quad (3)$$

So we define $I = \int N(x-y,t)f(y)dy$ and the stationary probability distribution becomes

$$p(N) = \frac{n \cdot \exp\left(\frac{1}{\sigma^2}\left(2r^*AN + 2r^*KN - r^*N^2 - \frac{2mI}{N} \right) \right)}{\sigma^2 N^{2+2(r^*AK+m)/\sigma^2}} \quad (4)$$

Without diffusion (m = 0) this distribution diverges at N=0 implying that the **population always become extinct. With diffusion neighbor locations can recover the local population from extinction** (provided local extinction is not very frequent). In particular, in the mean field limit l_m/l_e»1, we have

$$I = \int_0^\infty dN\, Np(N) \quad (5)$$

which transforms Eq. (4) into a implicit equation for p(N), with two non-zero solutions characterized by values I_1 and I_2 (with I_1<I_2)

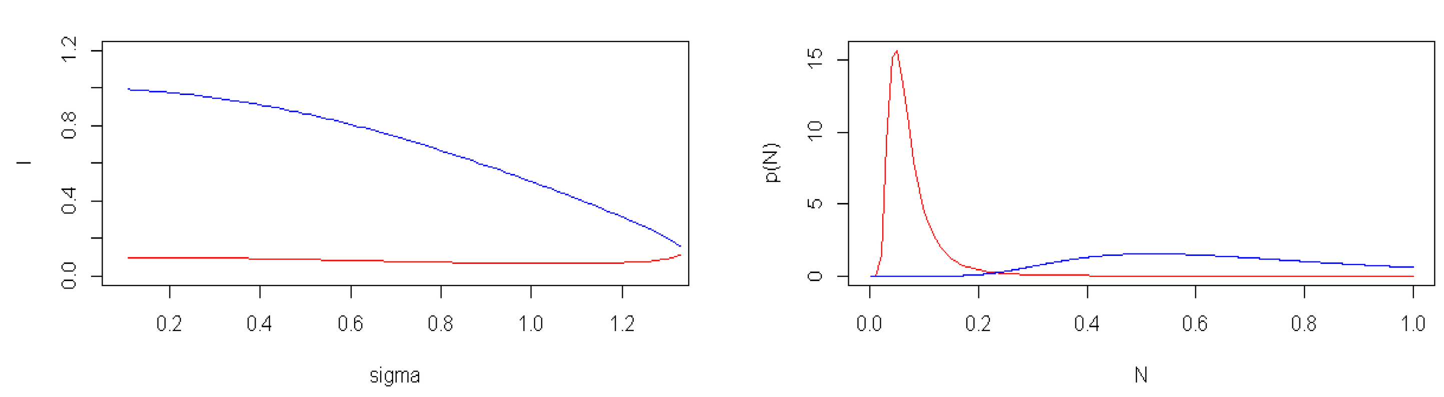

Fig 2. *I as a function of the environmental fluctuations σ (left): I_1 (red) and I_2 (blue) at the mean field approximation and K=1, A=0.1, and r = m = 1. Probability distributions as functions of population N at the mean field approximation (right): Here, I_1 = 0.075(red) and I_2 = 0.7 (blue) for the parameters K=1, A=0.1, r = m = 1, and σ = 0.75.*

Characterizing the extinction threshold

In the mean field approximation (**l_m/l_e»1**) the extinction threshold σ_{ext} is well estimated, by computing the value of the amplitude of environmental fluctuations σ that locates the maximum of $p_{I_2}(N)$ at the Allee threshold A.

$$\sigma_{ext} = \frac{\sqrt{Am(I_2 - A)}}{A} \quad (6)$$

This **extinction threshold decreases as diffusion becomes more local** $l_m \sim l_e$

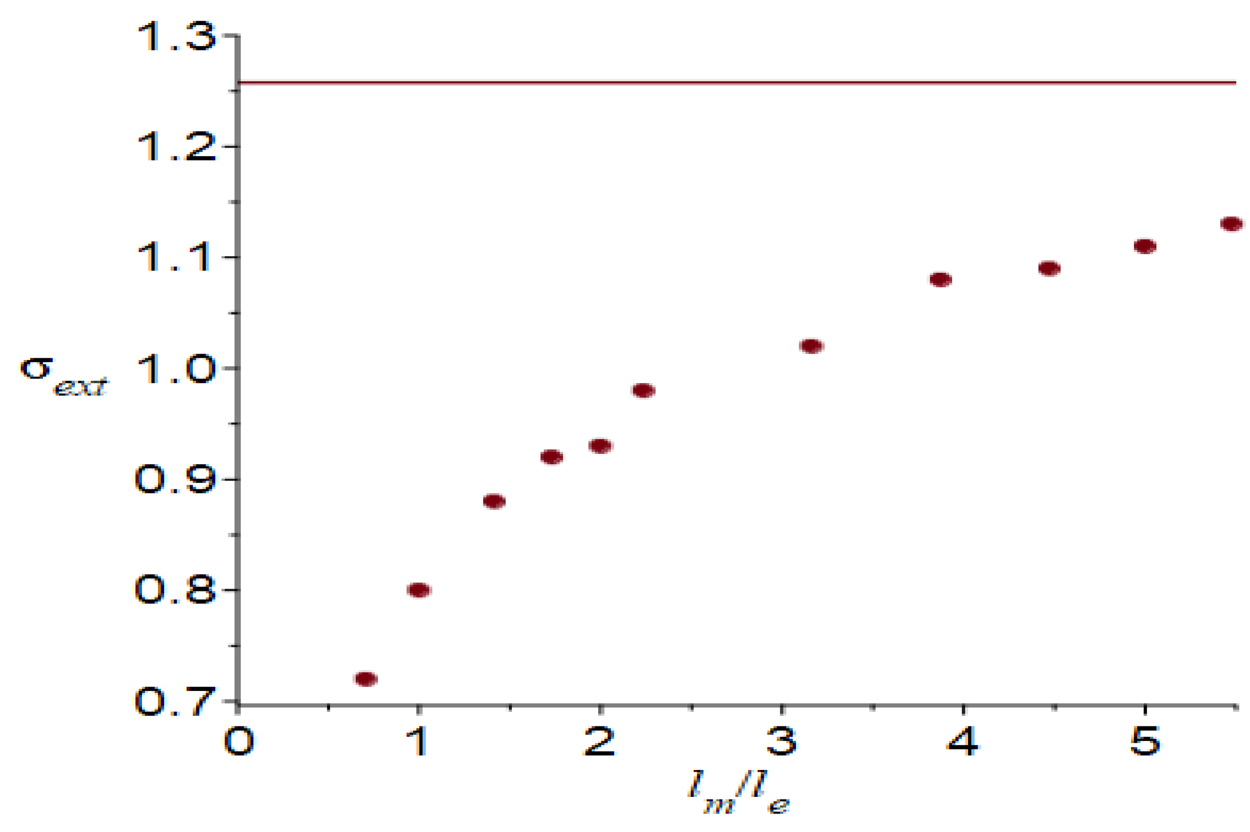

Fig 3. *Extinction threshold σ_{ext} as a function of relative diffusion distance l_m/l_e: Mean field approximation l_m»l_e (red solid line) and numerical results (red dots) for different values of l_m/l_e. Here, K=1, A = 0.1, and r=m= 1*

Decreasing the migration rates decreases the extinction threshold. The **extinction threshold** is further **decreased if** the **diffusion rate is decreased** or the **Allee threshold increased**.

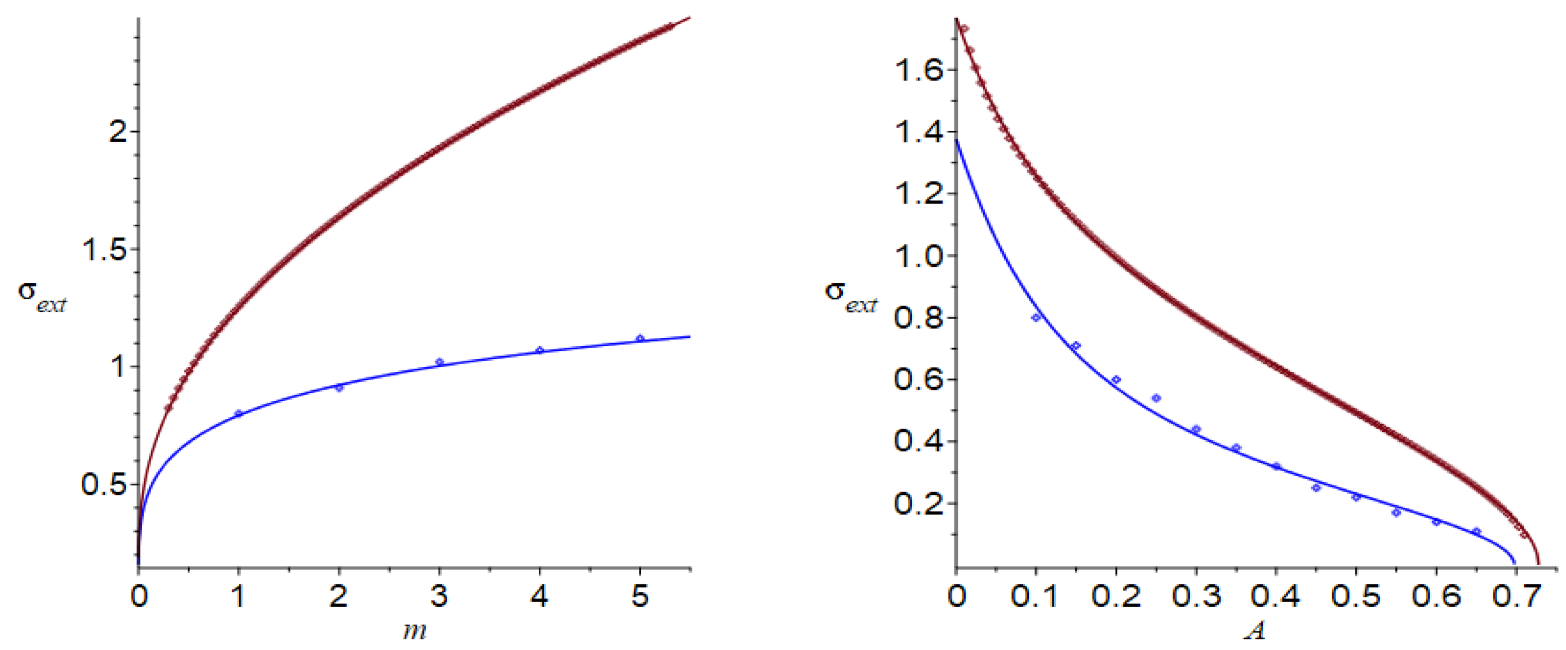

Fig 4. *Extinction threshold as a function of the migration rate m (left) and of the Allee threshold A (right): Mean field approximation l_m»l_e (red dots) and numerical results for local diffusion l_m=l_e=1. Lines show fits to the dots. Here, K=1, and r=1. Also, for the functions at the left we have A=0.1 as a parameter and for the functions at the right we have a fixed migration rate m=1.*

Conclusions

We have described the effects of relative diffusion length l_m/l_e, diffusion rate m, and minimum viable population A (Allee threshold) in the extinction threshold σ_{ext} for the amplitude of environmental fluctuations.

This provides a new tool for the assessment of extinction risk under the arrival of increased amplitude of environmental fluctuations for a spatially extended population.

Forthcoming Research

Effects of habitat fragmentation and interspecies interaction in the extinction threshold σ_{ext}.

References

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Contact information

Rodrigo Crespo (rodcresp@ucm.es), Francisco J. Cao (francao@fis.ucm.es)

Out of Equilibrium Dynamics Group (http://www.ucm.es/info/oeqdyn/)

Universidad Complutense de Madrid, Spain